BIHARMONIC ISOMETRIC IMMERSIONS INTO AND BIHARMONIC RIEMANNIAN SUBMERSIONS FROM $M^2 \times \mathbb{R}$

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Abstract

In this paper, we study biharmonic isometric immersions of a surface into and biharmonic Riemannian submersions from the product space $M^2 \times \mathbb{R}$. We give a classification of proper biharmonic isometric immersions of a surface with constant mean curvature into $M^2 \times \mathbb{R}$. More precisely, we prove that a surface with constant mean curvature H in $M^2 \times \mathbb{R}$ is proper biharmonic if and only if it is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right) \times \mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right) \times \mathbb{R}$. We also obtain a complete classification of proper biharmonic Hopf cylinders in $M^2 \times \mathbb{R}$. On the other hand, we give a classification of biharmonic (including harmonic) Riemannian submersions $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ from the product space, and construct many family of proper biharmonic Riemannian submersions $M^2 \times \mathbb{R} \to \mathbb{R}^2$.

1. Introduction and preliminaries

In this paper, we work in the category of smooth objects, so manifolds, maps, vector fields, etc, are assumed to be smooth unless it is stated otherwise.

As is well known, a *harmonic map* $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the energy functional

$$E(\varphi, \Omega) = \frac{1}{2}\int_{\Omega} |\mathrm{d}\varphi|^2 \,\mathrm{d}x.$$

The Euler-Lagrange equation is given by the vanishing of the tension filed $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ (see [2]), i.e., the map φ is harmonic if and only if $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi = 0$ holds identically.

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In 1983, J. Eells and L. Lemaire [14] first proposed studying k-polyharmonic maps (include biharmonic maps as a special cases). Let us recall the definition of biharmonic maps. If a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a critical point of the bienergy

$$E^2(\varphi, \Omega) = \frac{1}{2}\int_\Omega |\tau(\varphi)|^2 \, \mathrm{d}x$$

for every compact subset Ω of M, then we call it a *biharmonic map*, where $\tau(\varphi) = \mathrm{Trace}_g \nabla \mathrm{d}\varphi$ is the tension field of φ. In 1986, Jiang [19] first computed the first variation of the functional to find that φ is biharmonic if and only if its bitension field vanishes identically, i.e.,

$$\tau^2(\varphi) := \mathrm{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_{\nabla^M})\tau(\varphi) - \mathrm{Trace}_g R^N(\mathrm{d}\varphi, \tau(\varphi))\mathrm{d}\varphi = 0, \tag{1}$$

where R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z.$$

We call a submanifold a **biharmonic submanifold** if the isometric immersion that defines the submanifold is a biharmonic map. By definition, biharmonic maps include harmonic maps as special cases, and biharmonic submanifolds can be viewed as a generalization of the notion of minimal submanifolds (i.e., harmonic (minimal)isometric immersions). Analogously, a Riemannian submersion is called a biharmonic Riemannian submersion if the Riemannian submersion is a biharmonic map. We call nonharmonic biharmonic maps (respectively, submanifolds, Riemannian submersion) **proper biharmonic maps (respectively, submanifolds, Riemannian submersion)**.

It is very interesting and important to study the existence problem and classification problem which are two fundamental problems in the study of biharmonic maps. For the existence problem of biharmonic maps, it would be interesting to know if there exists a proper biharmonic map between given two "good" model spaces. (The so-called "good " spaces include space forms, or more general symmetric spaces, or homogeneous spaces, etc.). For the classification problem of biharmonic maps, we would especially like to classify all proper biharmonic maps between two model spaces where the existence is known. The following two classification problems are typical and challenging as follows:

Chen's conjecture [12, 13, 11]: any biharmonic submanifold in a Euclidean space $\mathbb{R}^n$ is minimal (i.e., harmonic).
The generalized Chen's conjecture: any biharmonic submanifold of a Riemannian manifold (N^n, h) with non positive curvature $Riem^N \leq 0$ is harmonic

(minimal) (see e.g., [6–13]).
Though the Chen's conjecture is also known to be true in some other cases (see, e.g., [21, 7, 28, 30, 16], it is still open for the general case. For the generalized Chen's conjecture, Ou and Tang ([29]) gave many counter examples in a Riemannian manifold of negative curvature. For some recent progress on biharmonic submanifolds, we refer the readers to [1],[4-13],[15-22],[25-31], etc., and the references therein.

Riemannian submersions can be viewed as the dual notion of isometric immersions (i.e., submanifolds). Based on this, it is interesting to study biharmonicity of Riemannian submersions. Biharmonic Riemannian submersions were first studied by Oniciuc [22] in 2002. In 2010, Wang and Ou [34] first introduced the so-called integrability data and used it to study biharmonicity of a Riemannian submersion from a generic 3-manifold, and obtain a complete classification of biharmonic Riemannian submersions from a 3-dimensional space form into a surface. In 2019, Akyol and Ou [1] studied biharmonicity of a general Riemannian submersion and obtained biharmonic equations for Riemannian submersions with one-dimensional fibers and Riemannian submersions with basic mean curvature vector fields of fibers. Moreover, the authors [1] first used the so-called integrability data to study biharmonic Riemannian submersions from $(n+1)$-dimensional spaces with one-dimensional fibers. In 2018, the author [31] studied biharmonicity a more general setting of Riemannian submersion with a S^1 fiber over a compact Riemannian manifold. In 2018, the authors in [18] studied generalized harmonic morphisms and obtained many examples of biharmonic Riemannian submersions which are maps between Riemannian manifolds that pull back local harmonic functions to local biharmonic functions.

Finally, we refer the readers to the following interesting results. In 2011, the authors [27] gave complete classifications of constant mean curvature proper biharmonic surfaces in Thurston's 3-dimensional geometries and BCV 3-spaces, and a complete classification of proper biharmonic Hopf cylinders in BCV 3-spaces. In 2023, the authors [36] gave a complete classification of biharmonic constant mean curvature (CMC) surfaces in 3-dimensional Berger sphere S^3_ε, and a complete classification of proper biharmonic Hopf cylinders in S^3_ε. On the other hand, in 2010, the authors [34] gave complete classifications of Riemannian submersions from a 3-dimensional space form. In 2023, the authors [35] classified all proper biharmonic Riemannian submersions from BCV 3-diemnsional spaces into a surface, and proved that a proper biharmonic Riemannian submersions from a BCV

3-space exists only in $H^2\times\mathbb{R}\to\mathbb{R}^2$, or, $\widetilde{SL}(2,\mathbb{R})\to\mathbb{R}^2$. In a recent paper [36], the authors also gave complete classifications of biharmonic Riemannian submersions from a 3-dimensional Berger sphere and proved that a biharrmonic Riemannian submersion from a 3 Berger 3-sphere S^3_ε into a surface has to be harmonic.

In this paper, we study biharmonic isometric immersions of a surface into $M^2\times\mathbb{R}$ and biharmonic Riemannian submersions from product spaces $M^2\times\mathbb{R}$ into a surface. We give a complete classification of proper biharmonic isometric immersions of a surface with constant mean curvature into product spaces $M^2\times\mathbb{R}$. More precisely, we prove that a proper biharmonic surface with constant mean curvature H in $M^2\times\mathbb{R}$ is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right)\times\mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right)\times\mathbb{R}$. We also obtain a complete classification of proper biharmonic Hopf cylinders in $M^2\times\mathbb{R}$. On the other hand, we give a complete classification of biharmonic (including harmonic) Riemannians submersions $\pi: M^2\times\mathbb{R}\to(N^2,h)$ from $M^2\times\mathbb{R}$ to a surface and construct many family of proper biharmonic Riemannian submersions $\pi: M^2\times\mathbb{R}\to\mathbb{R}^2$.

2. Biharmonic isometric immersions of a surface with constant mean curvature surfaces into $M^2\times\mathbb{R}$

Biharmonic surfaces in 3-dimensional space forms have been completely classified ([20], [11], [9], [10]). In [27], the authors classified all proper biharmonic CMC surfaces in 3-dimensional BCV spaces and Thurston's 3-dimensional geometries. Biharmonic CMC surfaces in $M^2(c)\times\mathbb{R}$, where $M^2(c)$ denotes a space form, has also been studied in [15] and [17]. In the recent paper [36], the authors give a complete classification of biharmonic constant mean curvature surfaces in 3-dimensional Berger sphere S^3_ε. In this section, we will study biharmonic isometric immersions of a surface with constant mean curvature surfaces into $M^2\times\mathbb{R}$ and proper biharmonic Hopf cylinders in $M^2\times\mathbb{R}$, where M^2 is a general 2-dimensional manifold.

It is well known that a 2-dimensional Riemannian manifold (M^2,g) can be locally expressed as $(\mathbb{R}^2, e^{2p(x,y)}dx^2+dy^2)$ with respect to local coordinates (x,y) in $\mathbb{R}^2$. Without loss of generality, locally, we identify product spaces $M^2\times\mathbb{R}$ with $(\mathbb{R}^3, e^{2p(x,y)}dx^2+dy^2+dz^2)$ with respect to local coordinates (x,y,z) in $\mathbb{R}^3$.

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \tag{2}$$

and the Riemannian and the Ricci curvatures:

$$\begin{aligned} &R(X,Y,Z,W)=g(R(Z,W)Y,X),\\ &\mathrm{Ric}(X,Y)=\mathrm{Trace}_g R=\sum_{i=1}^{3}R(Y,e_i,X,e_i)=\sum_{i=1}^{3}\langle R(X,e_i)e_i,Y\rangle. \end{aligned} \tag{3}$$

It is not difficult to check that $\{E_1=e^{-p(x,y)}\frac{\partial}{\partial x},\ E_2=\frac{\partial}{\partial y},\ E_3=\frac{\partial}{\partial y}\}$ form an orthonomal frame on $M^2\times\mathbb{R}=(\mathbb{R}^2\times\mathbb{R},e^{2p(x,y)}dx^2+dy^2+dz^2)$.
With respect to this frame , a straightforward computation shows that

$$[E_1,E_2]=p_yE_1,\quad \text{all other}\ \ [E_i,E_j]=0,\quad i,j=1,2,3. \tag{4}$$

Let ∇ denote the Levi-Civita connection of $M^2\times\mathbb{R}$, then we can check that

$$\nabla_{E_1}E_1=-p_yE_2,\ \nabla_{E_1}E_2=p_yE_1,\ \text{all other}\ \nabla_{E_i}E_j=0,\ i,\ j=1,2,3. \tag{5}$$

A further computation gives the possible nonzero components of the curvatures:

$$\begin{aligned} &R_{1212}=g(R(E_1,E_2)E_2,E_1)=-p_{yy}-p_y^2,\\ &\text{all other}\ R_{ijkl}=g(R(E_k,E_l)E_j,E_i)=0,\ \ i,j,k,l=1,2,3. \end{aligned} \tag{6}$$

and the Ricci curvature:

$$\mathrm{Ric}\,(E_1,E_1)=\mathrm{Ric}(E_2,E_2)=-p_{yy}-p_y^2,\ \text{all other}\ \mathrm{Ric}\,(E_i,E_j)=0,i,j=1,2,3. \tag{7}$$

The following equation for biharmonic hypersurfaces in a generic Riemannian manifold will be used in our study of biharmonic surface in the product space.

Theorem 2.1. ([26]) *Let $\varphi: M^m\longrightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta=H\xi$. Then φ is biharmonic if and only if:*

$$\begin{cases}\Delta H-H|A|^2+H\mathrm{Ric}^N(\xi,\xi)=0,\\ 2A\,(\mathrm{grad}\,H)+\frac{m}{2}\mathrm{grad}\,H^2-2\,H\,(\mathrm{Ric}^N\,(\xi))^{\top}=0,\end{cases} \tag{8}$$

where $\mathrm{Ric}^N: T_qN\longrightarrow T_qN$ denotes the Ricci operator of the ambient space defined by $\langle\mathrm{Ric}^N(Z),W\rangle=\mathrm{Ric}^N(Z,W)$ and A is the shape operator of the hypersurface with respect to the unit normal vector ξ.

We now study biharmonic surfaces with constant mean curvature H in the product spaces $M^2\times\mathbb{R}$.

Theorem 2.2. *A surface with constant mean curvature H in the product space $M^2\times\mathbb{R}$ is proper biharmonic if and only if it is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right)\times\mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right)\times\mathbb{R}$.*

Proof. Let $\{e_1=\sum_{i=1}^{3}a_1^iE_i,\ \ e_2=\sum_{i=1}^{3}a_2^iE_i,\ \ \xi=\sum_{i=1}^{3}a_3^iE_i\}$ be an adapted orthonormal frame with ξ being normal to the surface. By a simple computation, we use the Ricci curvature (7) to obtain $\mathrm{Ric}\,(\xi,\xi)=R\,(1-(a_3^3)^2)\,,\quad (\mathrm{Ric}\,(\xi))^\top=-Ra_3^3(a_1^3e_1+a_2^3e_2)$, where $R=-p_{yy}-p_y^2$. From these and using biharmonic surface equation (8), one finds that a surface with constant mean curvature H is biharmonic if and only if

$$-H\,[|A|^2-R\,(1-(a_3^3)^2)]=0,\ Ra_1^3a_3^3H=0,\ \text{and}\ Ra_2^3a_3^3H=0. \tag{9}$$

This has solution $H=0$ implying that the surface is minimal, or,

$$|A|^2=R\,(1-(a_3^3)^2)\,,\quad Ra_1^3a_3^3=0,\ \text{and}\ Ra_2^3a_3^3=0, \tag{10}$$

where $R=-p_{yy}-p_y^2$.
We solve Eq. (10) by considering the following cases:
Case I: $R=-p_{yy}-p_y^2=0$. In this case, we have $|A|^2=0$ and hence $H=0$ meaning that the surface is minimal.
Case II: $R=-p_{yy}-p_y^2\neq 0$. In this case, by the last two equations of (10), we have either $a_3^3=0$ or $a_1^3=a_2^3=0$. This is discussed by the following two cases.
For Case II-A: $a_3^3=0$, applying the first equation of (10) we conclude that

$$|A|^2=R. \tag{11}$$

Clearly, $a_3^3=0$ means that the normal vector field of the surface Σ is of the form $\xi=a_1^1E_1+a_1^2E_2$. This implies that the normal vector field ξ is always orthogonal to $E_3=\frac{\partial}{\partial z}$. Consequently, we can choose an another orthonormal frame $\{e_1=aE_1+bE_2, e_2=E_3, \xi=bE_1-aE_2\}$ adapted to the surface with $a^2+b^2=1$ and ξ being the unit normal vector filed. By a straightforward computation using (5) we have

$$\nabla_{e_1}\xi=\{ae_1(b)-be_1(a)-ap_y\}e_1,\ \nabla_{e_2}\xi=\left(ae_2(b)-be_2(a)\right)e_1. \tag{12}$$

With respect to the chosen adapted orthonormal frame, a further calculation gives the second fundamental form of the surface Σ as

$$\begin{aligned} &h(e_1,e_1)=-\langle\nabla_{e_1}\xi,e_1\rangle=-\{ae_1(b)-be_1(a)-ap_y\},\ h(e_1,e_2)=-\langle\nabla_{e_1}\xi,e_2\rangle=0,\\ &h(e_2,e_1)=-\langle\nabla_{e_2}\xi,e_1\rangle=-\left(ae_2(b)-be_2(a)\right),\ h(e_2,e_2)=-\langle\nabla_{e_2}\xi,e_2\rangle=0.\end{aligned} \tag{13}$$

It follows from (13), the symmetry $h(e_1,e_2)=h(e_2,e_1)=0$ that $e_2(b)=e_2(a)=0$, $h(e_1,e_1)=2H$, and hence $|A|^2=4H^2$. From this and using (11), we have

$$R(E_1,E_2,E_1,E_2)=-p_{yy}-p_y^2=R=4H^2. \tag{14}$$

This implies that M^2 has constant Gauss curvature $K^M=4H^2>0$. It follows from a well-known fact in the differential geometry of surfaces that M^2

can be identified with 2-sphere $S^2\left(\frac{1}{2|H|}\right)$ with positive constant Gauss curvature $c = 4H^2$. Therefore, it follows from Theorem 2.1 in [27] that a constant mean curvature surface in $S^2\left(\frac{1}{2|H|}\right) \times \mathbb{R}$ is proper biharmonic if and only if it is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right) \times \mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right) \times \mathbb{R}$.

For Case II-B: $a_1^3 = a_2^3 = 0$ and hence $a_3^3 = \pm 1$. It follows that $\mathrm{Span}\{e_1, e_2\} = \mathrm{Span}\{E_1, E_2\}$. This means that the surface is M^2 in $M^2 \times \mathbb{R}$, which is a minimal isometric immersion .
From which, the theorem follows. □

Theorem 2.3. *Let $\pi : M^2 \times \mathbb{R} \to M^2$ be a Riemannian submersion with $\pi(x, y, z) = (x, y)$ defined by the projection onto its first factor and $\beta : I \to M^2$ be an immersed regular curve parametrized by arc length. Then the Hopf cylinder $\Sigma = \cup_{s\in I}\pi^{-1}(\beta(s))$ is a proper biharmonic surface in $M^2 \times \mathbb{R}$ if and only if it is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right) \times \mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right) \times \mathbb{R}$ with nonzero constant mean curvature H. Furthermore, this holds implying that $M^2 \times \mathbb{R}$ has to be $S^2\left(\frac{1}{2|H|}\right) \times \mathbb{R}$, i.e., M^2 identify with a 2-sphere $S^2\left(\frac{1}{2|H|}\right)$ with positive constant Gauss curvature $c = 4H^2$.*

Proof. Consider the Riemannian submersion $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to M^2 = (\mathbb{R}^2, e^{2p(x,y)}dx^2 + dy^2)$ with $\pi(x, y, z) = (x, y)$ and let $\beta : I \to M^2 = (\mathbb{R}^2, e^{2p(x,y)}dx^2 + dy^2)$, $\beta(s) = (x(s), y(s))$ be an immersed regular curve parametrized by arc length with the geodesic curvature k_g. We take the horizontal lifts of the tangent and the principal normal vectors of the curve $\beta : X = aE_1 + bE_2$ and $\xi = bE_1 - aE_2$ (where $a = e^{p(x,y)}x'$, $b = y'$ and $a^2 + b^2 = 1$) together with $V = E_3$ to be an orthonormal frame adapted to the Hopf cylinder (see also [26]). A direct computation using (7) gives:

$$\mathrm{Ric}\,(\xi, \xi) = R = -p_{yy} - p_y^2,\ \mathrm{Ric}\,(\xi, X) = 0,\ \mathrm{Ric}\,(\xi, V) = 0. \tag{15}$$

We can check that the the geodesic torsion of the lifting curve $\pi^{-1}(\beta(s))$

$$\tau_g = -\langle \nabla_X V, \xi\rangle = -\langle \nabla_{aE_1+bE_2} E_3, bE_1 - aE_2\rangle = 0. \tag{16}$$

It follows from Eq. (16) in [26] that the surface $\Sigma = \cup_{s\in I}\pi^{-1}(\beta(s))$ in $M^2\times\mathbb{R}$ is biharmonic if and only if

$$\begin{cases} k_g'' - k_g(k_g^2 + 2\tau_g^2) + k_g\mathrm{Ric}(\xi,\xi) = 0,\\ 3k_g'k_g - 2k_g\mathrm{Ric}(\xi,X) = 0,\\ k_g'\tau_g + k\mathrm{Ric}(\xi,V) = 0.\end{cases}$$

We substitute (15) and (16) into the above equation to get

$$k_g'' - k_g^3 + k_gR = 0, \text{ and } 3k_g'k_g = 0, \tag{17}$$

where $R = -p_{yy} - p_y^2$.
Solving (17) we have $k_g = 0$ implying that the surface Σ is minimal surface, or β has constant geodesic curvature k_g and $k_g^2 = R = -p_{yy} - p_y^2 > 0$. We use the fact from [26] (Page 229) to conclude that the mean curvature of the Hopf cylinder given by $H = \frac{k_g}{2}$, and hence $|A|^2 = k_g^2 + 2\tau_g^2 = k_g^2 = R = -p_{yy} - p_y^2$. From these we conclude that the Hopf cylinder $\Sigma = \cup_{s\in I}\pi^{-1}(\beta(s))$ is proper biharmonic if and only if
(18)

$$H^2 = \frac{R}{4} = \frac{-p_{yy}-p_y^2}{4} = constant > 0,\ |A|^2 = R = -p_{yy} - p_y^2 = constant > 0.$$

We apply the characterizations of Hopf cylinders in $M^2\times\mathbb{R}$ given in Theorem 2.2 to obtain the Theorem.

□

Corollary 2.4. *A totally umbilical surface in $M^2\times\mathbb{R}$ is biharmonic if and only if it is minimal.*

Proof. Since a totally umbilical biharmonic surface in 3-dimensional Riemannian manifolds must have constant mean curvature H ([27]), we have from Theorem 2.2 to conclude the only potential totally umbilical proper biharmonic surface is a part of $S^1\left(\frac{1}{2\sqrt{2}|H|}\right)\times\mathbb{R}$ in $S^2\left(\frac{1}{2|H|}\right)\times\mathbb{R}$. Clearly, it is not totally umbilical. □

3. Harmonic and biharmonic Riemannian submersions from $M^2\times\mathbb{R}$

In this section, we study biharmonicity (including harmonicity) of submersions from $M^2\times\mathbb{R}\to(N^2,h)$ and also derive a classification of biharmonic (including harmonic) Riemannian submersions from $M^2\times\mathbb{R}$ by identifying $M^2\times\mathbb{R}$ with $(\mathbb{R}^3, e^{2p(x,y)}dx^2+dy^2+dz^2)$ in local coordinates.

Let $\pi : M^2\times\mathbb{R}\to(N^2,h)$ be a Riemannian submersion from $M^2\times\mathbb{R}$ to a surface with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$ and e_3 being vertical. By a computation similar to Remark 1 in [35], we have the following (19)–(25) (see

[35] for details).
The Lie brackets given by

$$[e_1,e_3]=f_3e_2+\kappa_1e_3,\ [e_2,e_3]=-f_3e_1+\kappa_2e_3,\ [e_1,e_2]=f_1e_1+f_2e_2-2\sigma e_3. \tag{19}$$

where $\{f_1,\ f_2,f_3,\kappa_1,\ \kappa_2,\ \sigma\}$ is the (generalized) integrability data. Moreover, $f_3=0$ if and only if the above frame is adapted to π.
The Levi-Civita connection can be given by
(20)

$$\begin{aligned}&\nabla_{e_1}e_1=-f_1e_2,\ \ \nabla_{e_1}e_2=f_1e_1-\sigma e_3,\ \ \nabla_{e_1}e_3=\sigma e_2,\\ &\nabla_{e_2}e_1=-f_2e_2+\sigma e_3,\ \ \nabla_{e_2}e_2=f_2e_1,\ \ \nabla_{e_2}e_3=-\sigma e_1,\\ &\nabla_{e_3}e_1=-\kappa_1e_3+(\sigma-f_3)e_2,\nabla_{e_3}e_2=-(\sigma-f_3)e_1-\kappa_2e_3,\nabla_{e_3}e_3=\kappa_1e_1+\kappa_2e_2,\end{aligned}$$

We denote by $e_i=\sum\limits_{j=1}^{3}a_i^jE_j,\ i=1,2,3$ and use (5), (6) and (20) to check that the Jacobi identities as

$$\begin{aligned}&e_3(f_1)+(\kappa_1+f_2)f_3-e_1(f_3)=0,\ e_3(f_2)+(\kappa_2-f_1)f_3-e_2(f_3)=0,\\ &2e_3(\sigma)+\kappa_1f_1+\kappa_2f_2+e_2(\kappa_1)-e_1(\kappa_2)=0,\end{aligned} \tag{21}$$

and the terms of the curvature tension as follows

$$\begin{cases}R^M(e_1,e_3,e_1,e_2)=-e_1(\sigma)+2\kappa_1\sigma=-a_2^3a_3^3R,\\ R^M(e_1,e_3,e_1,e_3)=e_1(\kappa_1)+\sigma^2-\kappa_1^2+\kappa_2f_1=(a_2^3)^2R,\\ R^M(e_1,e_3,e_2,e_3)=e_1(\kappa_2)-e_3(\sigma)-\kappa_1f_1-\kappa_1\kappa_2=-a_1^3a_2^3R,\\ R^M(e_1,e_2,e_1,e_2)=e_1(f_2)+e_2(f_1)-f_1^2-f_2^2+2f_3\sigma-3\sigma^2=(a_3^3)^2R,\\ R^M(e_1,e_2,e_2,e_3)=-e_2(\sigma)+2\kappa_2\sigma=a_1^3a_3^3R,\\ R^M(e_2,e_3,e_1,e_3)=e_2(\kappa_1)+e_3(\sigma)+\kappa_2f_2-\kappa_1\kappa_2=-a_1^3a_2^3R,\\ R^M(e_2,e_3,e_2,e_3)=\sigma^2+e_2(\kappa_2)-\kappa_1f_2-\kappa_2^2=(a_1^3)^2R,\end{cases} \tag{22}$$

where $R=-p_{yy}-p_y^2$.
We have Gauss curvature of the base space as follows

$$K^N=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma. \tag{23}$$

Moreover,

$$e_3(K^N)=e_3\{e_1(f_2)-e_2(f_1)-f_1^2-f_2^2+2f_3\sigma\}=0. \tag{24}$$

When $f_3=0$, Gauss curvature of the base space becomes

$$K^N=e_1(f_2)-e_2(f_1)-f_1^2-f_2^2. \tag{25}$$

3.1. **Harmonic Riemannian submersions from $M^2 \times \mathbb{R}$.** It is a well known fact that harmonic maps are always biharmonic maps. Therefore, it would be interesting to study harmonicity of Riemannian submersions from $M^2 \times \mathbb{R}$ before studying biharmonicity of them. In [37], the authors classified all harmonic Riemannian submersions from Thuston's 3-spaces, BCV 3-spaces and Berger 3-sphere. In this subsection, we will study harmonic Riemannian submersions from $M^2\times\mathbb{R}$. In next subsection, we we will classify all proper biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$.

Now we are going to give the following classification of harmonic Riemannian submersions from $M^2 \times \mathbb{R}$.

Theorem 3.1. *A Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ is harmonic if and only if it is the projection $\pi : M^2 \times \mathbb{R} \to M^2$.*

Proof. Let $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ be a Riemannian submersion with an orthonormal frame $\{e_1,\ e_2,\ e_3\}$, e_3 being vertical, and the (generalized) integrability data $\{f_1, f_2, f_3, \kappa_1, \kappa_2, \sigma\}$. Denoting by $e_i = \sum_{j=1}^{3} a_i^j E_j, i = 1, 2, 3$. It follows from Proposition 2.2 in [37] and (22) that if a Riemannian submersion $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^2 \times \mathbb{R}, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (N^2, h)$ is harmonic, then we have

$$(26)\qquad \begin{cases} -e_1(\sigma) = -a_2^3 a_3^3 R,\\ \sigma^2 = (a_2^3)^2 R,\\ -e_3(\sigma) = -a_1^3 a_2^3 R = 0,\\ K^N = 3\sigma^2 + (a_3^3)^2 R,\\ -e_2(\sigma) = a_1^3 a_3^3 R,\\ e_3(\sigma) = -a_1^3 a_2^3 R = 0,\\ \sigma^2 = (a_1^3)^2 R, \end{cases}$$

where $R = -p_{yy} - p_y^2$ and $K^N = e_1(f_2) + e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma$.

Since Gauss curvature of M^2 equals $R = -p_{yy} - p_y^2$, it follows that if $R = -p_{yy} - p_y^2 = 0$ then M^2 is a Euclidean space $\mathbb{R}^2$, i.e., the product space $M^2 \times \mathbb{R}$ is actually a Euclidean space $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. On the other hand, we use the 2nd equation and the 4th equation of (26) to conclude that $\sigma = 0$ and hence Gauss curvature of the base space $K^N = 0$, which implies that the base space N^2 can be viewed as a 2-dimensional Euclidean space $\mathbb{R}^2$. In the case, a harmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ from a product space $M^2 \times \mathbb{R}$ exists

only in the orthogonal projections $\pi : \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$.

From now on, suppose $R = -p_{yy} - p_y^2 \neq 0$. In this case, we first use the 2nd equation and the 7th equation of (26) to have $(a_1^3)^2 = (a_2^3)^2$. On the other hand, using the 3rd equation and the 6th equation of (26) we have $a_1^3 a_2^3 = 0$. From these, we have $a_1^3 = a_2^3 = 0$ and hence $(a_3^3)^2 = 1$. We apply the 2nd and the 4th equation of (26) to get $\sigma^2 = 0$ and hence Gauss curvature of the base space $K^N = R$ equals Gauss curvature of M^2. This implies that N^2 is locally isometric to M^2. Since $(a_3^3)^2 = 1$, one finds that the orthonormal frame $\{e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3\}$, together with (5), is an adapted frame to the Riemannian submersion π with $e_3 = E_3$ being vertical. Combining these, a harmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ is actually the projection $\pi : M^2 \times \mathbb{R} \to M^2$.
From which the theorem follows. □

3.2. **Biharmonic Riemannian submersions from the product spaces $M^2 \times \mathbb{R}$.** In this subsection, we will classify all proper biharmonic Riemannian submersions from $M^2 \times \mathbb{R}$, which are not harmonic.
The following lemma was obtained in [34] which will be later used in the rest of the paper.

Lemma 3.2. *([34]) Let $\pi : (M^3, g) \longrightarrow (N^2, h)$ be a Riemannian submersion with the adapted frame $\{e_1,\ e_2,\ e_3\}$ and the integrability data $f_1, f_2, \kappa_1,\ \kappa_2$ and σ. Then, the Riemannian submersion π is biharmonic if and only if*
(27)
$$\begin{cases} -\Delta^M \kappa_1 - 2\sum\limits_{i=1}^{2} f_i e_i(\kappa_2) - \kappa_2 \sum\limits_{i=1}^{2} (e_i(f_i) - \kappa_i f_i) + \kappa_1 \left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0, \\ -\Delta^M \kappa_2 + 2\sum\limits_{i=1}^{2} f_i e_i(\kappa_1) + \kappa_1 \sum\limits_{i=1}^{2} (e_i(f_i) - \kappa_i f_i) + \kappa_2 \left(-K^N + \sum\limits_{i=1}^{2} f_i^2\right) = 0, \end{cases}$$
where $K^N = R^N_{1212} \circ \pi = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2$ is Gauss curvature of Riemannian manifold (N^2, h).

The following proposition was derived in [35]

Proposition 3.3. *(see [35]) Let $\pi : (M^3, g) \to (N^2, h)$ be a Riemannian submersion from 3-manifolds with an orthonormal frame $\{e_1, e_2, e_3\}$ and e_3 being vertical. If $\nabla_{e_1} e_1 = 0$, then either $\nabla_{e_2} e_2 = 0$; or $\nabla_{e_2} e_2 \not\equiv 0$, and the frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π.*

We apply Proposition 3.3 to have the following theorem which will be used to prove our main result

Theorem 3.4. *Let $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (N^2, h)$ be a Riemannian submersion from product spaces with $R = -p_{yy} - p_y^2 \neq 0$. Then, there exists such an adapted frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2,\ e_2,\ e_3\}$ of the Riemannian submersion π with e_3 being vertical. Moreover, if E_3 is not vertical, then $\nabla_{e_2} e_2 \neq 0$, i.e., $f_2 \neq 0$.*

Proof. Clearly, if E_3 is tangent to the fiber of the Riemannian submersion π, i.e., e_3 is perpendicular to both E_1, E_2, then any basic vector field can be expressed as the form $e = a^2 E_1 + b^2 E_2$, and $a^2 + b^2 = 1$.

From now on, we need only to suppose that E_3 is not vertical, i.e., e_3 is not parallel to E_3. Then, the vector filed $e_1 = e_3 \times E_3$ is horizontal and hence $\langle e_1, E_3 \rangle = 0$. From this, one can obtain a defined orthonormal frame $\{e_1,\ e_2 = e_3 \times e_1,\ e_3\}$ on M^3. If denoting by $e_i = \sum\limits_{j=1}^{3} a_i^j E_j, i = 1, 2, 3,$ together with $\langle e_1, E_3 \rangle = 0$, then the vector horizontal filed e_1 can be expressed as the form $e_1 = a_1^1 E_1 + a_1^2 E_2$ and hence $a_1^3 = 0$, $(a_1^1)^2 + (a_1^2)^2 = 1$. From these, we have the following

$$a_1^3 = 0,\ a_3^3 \neq \pm 1 \text{ and } a_2^3 \neq 0. \tag{28}$$

Moreover, one has also the following equalities as

$$f_1 = 0,\ \nabla_{e_1} e_1 = 0. \tag{29}$$

In fact, by a straightforward computation we get

$$\nabla_{e_1} e_1 = \nabla_{e_1} (\sum_{i=1}^{3} a_1^i E_i) = \sum_{i=1}^{3} e_1(a_1^i) E_i + \sum_{i,j=1}^{3} a_1^j a_1^i \nabla_{E_j} E_i. \tag{30}$$

In addition, using(20), the above has another expression as

$$\nabla_{e_1} e_1 = -f_1 e_2 = -f_1 \sum_{i=1}^{3} a_2^i E_i. \tag{31}$$

Equating (30) and (31) and making a comparison of the coefficient of E_3, we obatin

$$\begin{aligned} -f_1 a_2^3 &= \langle -f_1 \sum_{i=1}^{3} a_1^i E_i, E_3 \rangle = \langle \nabla_{e_1} e_1, E_3 \rangle \\ &= \langle \sum_{i=1}^{3} e_1(a_1^i) E_i + \sum_{i,j=1}^{3} a_1^j a_1^i \nabla_{E_j} E_i, E_3 \rangle = e_1(a_1^3) = 0, \end{aligned} \tag{32}$$

which has been used (5) and $a_1^3 = 0$. This deduces $f_1 = 0$ for $a_2^3 \neq 0$, from which we get (29).

Applying (5), (20) and $a_1^3 = f_1 = 0$ and performing a computation in the same way to those used computing (30)–(32), we have

$$\begin{cases} e_1(a_2^3) = -\sigma a_3^3, \\ e_1(a_3^3) = \sigma a_2^3, \\ e_2(a_2^3) = 0, \\ e_2(a_3^3) = 0, \\ e_3(a_2^3) = -\kappa_2 a_3^3, \\ e_3(a_3^3) = \kappa_2 a_2^3, \\ \kappa_1 a_3^3 = (\sigma - f_3) a_2^3, \\ f_2 a_2^3 = \sigma a_3^3. \end{cases} \tag{33}$$

Since $\nabla_{e_1} e_1 = 0$, we apply Lemma 3.3 to have either $\nabla_{e_2} e_2 \not\equiv 0$, and the frame $\{e_1, e_2, e_3\}$ is adapted to the Riemannian submersion π, or $\nabla_{e_2} e_2 = 0$. For the latter case: $\nabla_{e_2} e_2 = 0$, i.e., $f_2 = 0$, we will show that if $a_3^3 = 0$, then the chosen frame $\{e_1, e_2, e_3\}$ is also adapted to the Riemannian submersion π whilst the other case $a_3^3 \neq 0, \pm 1$ and $f_2 = 0$ can not happen.

In fact, first of all, we show that the case $a_3^3 \neq 0, \pm 1$ and $f_2 = 0$ can not happen. If otherwise, substituting $f_2 = 0$ into the 8th equation of (33), together with $a_2^3 \neq 0$ and $a_3^3 \neq 0, \pm 1$, we get $\sigma = 0$. Furthermore, substituting these and $f_1 = f_2 = 0$ into the 4th equation of (22), we have $R = -p_{yy} - p_y^2 = 0$ contradicting the assumption $R \neq 0$. Secondly, we now consider the case $f_2 = a_3^3 = 0$. Since $a_3^3 = a_1^3 = 0$, it is very easy to see that $(a_2^3)^2$ equals 1. We apply the 2nd equation and the 7th equation of (33) separately to obtain $\sigma = 0$ and hence $f_3 = 0$. Combining these, for the case $a_3^3 = 0$ and $a_1^3 = f_1 = f_2 = 0$, the frame $\{e_1, e_2, e_3\}$ is an adapted frame of the Riemannian submersion.
From which, we obtain the theorem. □

Remark 1. Let $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)} dx^2 + dy^2 + dz^2) \to (N^2, h)$ be a Riemannian submersion. Then, we have the following cases:

(a) If $R = -p_{yy} - p_y^2 = 0$, then the potential product space is $\mathbb{R}^3$. We apply Theorem 3.3 in [34] to conclude that biharmonic Riemannian submersion $\pi : \mathbb{R}^3 \to (N^2, h)$ has to be harmonic.

(b) If $(a_3^3)^2 = 1$, i.e., E_3 is vertical, then we use (5) to obtain the tension of the Riemannian submersion $\tau(\pi)$ vanishes, i.e., π is harmonic.

Remark 2. Let $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (N^2, h)$ be a Riemannian submersion with e_3 being vertical. Suppose that E_3 is not vertical (i.e., $a_3^3 \neq \pm 1$) and $R = -p_{yy} - p_y^2 \neq 0$. From the proof of Theorem 3.4, we can choose such an orthonormal frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2, e_2, e_3\}$ on $M^2 \times \mathbb{R}$ with $a_1^3 = f_1 = 0$, $a_3^3 \neq \pm 1$ and $a_2^3 \neq 0$. Applying Theorem 3.4 and together with the proof of Theorem 3.4, one can further find the following:

(i) $f_2 \neq 0$. In this case, the chosen frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2, e_2, e_3\}$ is actually adapted to the Riemannian submersion π with $a_1^3 = f_1 = f_3 = 0$.

(ii) the case $f_2 = 0$ and $a_3^3 \neq 0, \pm 1$ can not happen.

(iii) $f_2 = 0$ and $a_3^3 = 0$ and hence $a_2^3 = \pm 1$. We show that Riemannian submersions π must be $M^2 \times \mathbb{R} \to \mathbb{R}^2$. In fact, note first that the chosen frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2, e_2, e_3\}$ is adapted to the Riemannian submersion π with $a_1^3 = f_1 = f_3 = 0$. For $a_3^3 = a_1^3 = f_1 = f_2 = f_3 = 0$ and $a_2^3 = \pm 1$, by a simple computation using the 3rd, the 6th and the 7th equation of (33) gives $\sigma = \kappa_2 = 0$. Moreover, from these, we use the 2th equation of (22) to conclude $\kappa_1 \neq 0$. In addition, it is easy to find from (23) that Gauss curvature of the base space $K^N = e_1(f_2) - e_2(f_1) - f_1^2 - f_2^2 + 2f_3\sigma = 0$ and hence N^2 is a Euclidean space $\mathbb{R}^2$. Therefore, Riemannian submersions π must be $M^2 \times \mathbb{R} \to \mathbb{R}^2$.

(iv) On the condition of (iii), there exist many family of proper biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$. Moreover, with respect to local coordinates, a proper biharmonic Riemannian submersion $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$ can be locally expressed as

$$\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z),$$

and the function $p(x, y)$ satisfies PDE (90) and $p_y \neq 0$.

In fact, it is observed that the orthonomal frame $\{e_1 = a_1^1 E_1 + a_1^2 E_2,\ e_2 = E_3,\ e_3 = a_3^1 E_1 + a_3^2 E_2\}$ on $M^2 \times \mathbb{R}$ is an adapted frame to the Riemannian submersion π with e_3 being vertical. Note also that the basic vector field $e_1 = a_1^1 E_1 + a_1^2 E_2$ and the vertical field $e_3 = a_3^1 E_1 + a_3^2 E_2$ are tangent to M^2 and another basic the vector field $e_2 = E_3$ is perpendicular to M^2 (i.e., $e_2 = E_3$ parallel to $\mathbb{R}$). Since $\nabla_{e_1} e_1 = 0$, it is easy to see that an arbitrary integral curve of e_1 is geodesics on $M^2 \subset M^2 \times \mathbb{R}$. It follows from a well-known fact in the differential geometry of surfaces that one can parametrize M^2 by $r = r(u, v)$ so that the $v-$curves are the integral curves of e_1 and the $u-$curves are the orthogonal trajectories

of the $v-$curves, and the Riemannian metric on M^2 can be also expressed as $g = e^{2q(u,v)}du^2 + dv^2$ by reparametrization with $e_3 = e^{-q(u,v)}\frac{\partial}{\partial u}$, $e_1 = \frac{\partial}{\partial v}$ and hence $R = -q_{vv} - q_v^2 = -p_{yy} - p_y^2$ on M^2. It turns out that with respect to local coordinates (u, v, z), the Riemannian submersion $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$ can be locally expressed as

$$\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2q(u,v)}du^2 + dv^2 + dz^2) \to (\mathbb{R}^2, dv^2 + dz^2),\ \pi(u, v, z) = (v, z).$$

Since the Coordinate transformation $(x, y) \to (u, v)$ is an isometric transformation $M^2 \to M^2$ and it is a convention to use coordinates (x, y) in place of (u, v), the above Riemannian submersion $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$ can be also locally expressed as

$$\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z).$$

In addition, by Proposition 3.7, Remark 3 and Example 1, one sees that there exist many family of proper biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to \mathbb{R}^2$, where $M^2 \times \mathbb{R}$ satisfies certain conditions determined by $p(x, y)$ solving (90).

We now give a characterization of Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$.

Theorem 3.5. *If $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ is a proper biharmonic Riemannian submersion from the product space, then*
(a) The target surface is flat, and locally the map can be expressed as

$$\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z),$$

where $p(x, y)$ is a function solving PDE (90) and $p_y \neq 0$, or,
(b) With respect to another local coordinates, it is a Riemannian submersion

$$\begin{aligned} &\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2Q(v)}d\mu^2 + dv^2 + dz^2) \to (\mathbb{R}^2, dv^2 + e^{2\lambda(v)}d\phi^2),\\ &\pi(\mu, v, z) = (v, \phi) = (v, \mp A\mu \pm Az + B), \end{aligned} \tag{34}$$

where $Q(v) = \ln|\tan\alpha(v)|$, $\lambda = \ln|\sin\alpha(v)| - \ln A$ depend on only v , B and $A > 0$ are constants, and a nonconstant function $\alpha(v)$ solves the following ODE

$$\alpha''' \sin\alpha\cos^2\alpha + \cos\alpha(\sin^2\alpha + 3)\alpha'\alpha'' + \sin\alpha(2\cos^2\alpha + 3)\alpha'^3 = 0. \tag{35}$$

Proof. For the proof of the theorem, we will use a local expression of $M^2 \times \mathbb{R}$ with the form $(\mathbb{R}^2 \times \mathbb{R}, e^{2p(x,y)}dx^2 + dy^2 + dz^2)$ in local coordinates, where M^2 has such a local expression the form $(\mathbb{R}^2, e^{2p(x,y)}dx^2 + dy^2)$. Let ∇ denote the Levi-Civita connection on $M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2)$ and by $e_i = \sum_{j=1}^{3} a_i^j E_j$, $i = 1, 2, 3$. For considering proper biharmonic Riemannian submersions $\pi : M^2 \times \mathbb{R} \to$

(N^2, h), by Remark 1, we need only to consider $R = -p_{yy} - p_y^2 = 0 \neq 0$ and $a_3^3 \neq \pm 1$ in the rest of the proof of the theorem. Therefore, by Theorem 3.4, there is an adapted frame $\{e_1,\ e_2,\ e_3\}$ to the Riemannian submersion π with $e_1 = a_1^1 E_1 + a_1^2 E_2$, e_3 being vertical, $a_1^3 = f_1 = f_3 = 0$, and hence $e_3(f_1) = e_3(f_2) = 0$ (see [34]).

By Remark 2, we now just need to consider the following two cases:
Case I: $a_3^3 = 0$. In this case, we have $a_2^3 = \pm 1$ since $a_1^3 = 0$. Noting that $f_3 = 0$, we use the 8th equations of (33) to have $f_2 = 0$. By (iii) and (iv) of Remark 2, we immediately give the statement (a).

Case II: $a_3^3 \neq 0, \pm 1$. This means $a_2^3 \neq 0, \pm 1$ since $a_1^3 = 0$. In this case, we show that for $f_2 \neq 0$, there exist also proper biharmonic Riemannian submersions from certain product spaces $M^2 \times \mathbb{R}$.
By Remark 2 and combining the above, we have the following conditions:

$$a_2^3,\ a_3^3 \neq 0, \pm 1,\ f_2 \neq 0,\ a_1^3 = f_1 = f_3 = 0,\ e_3(f_1) = e_3(f_2) = 0. \tag{36}$$

This case corresponds to the statement (b) which is obtained by the following two steps:

Step 1: show that $e_2(f_2) = e_2(\kappa_1) = e_3(\kappa_1) = e_2(\sigma) = e_3(\sigma) = e_2(R) = e_3(R) = \kappa_2 = 0,\ \kappa_1\sigma \neq 0$.

Firstly, we have $\sigma \neq 0$ by the 8th equation of (33).

Secondly, show that $\kappa_2 = 0$ and hence $e_2(f_2) = e_3(f_2) = e_2(\sigma) = e_3(\sigma) = e_2(\kappa_1) = e_3(\kappa_1) = e_3(R) = e_2(R) = 0$.

We use the 5th and the 6th equation of (33), (24) and (36) and apply e_3 to both sides of the 8th equation of (33) to get

$$-f_2\kappa_2 a_3^3 = e_3(\sigma)a_3^3 + \sigma\kappa_2 a_2^3. \tag{37}$$

One substitutes the 8th equation of (33) into (37), together with (36), and simplifies the resulting equation to have

$$e_3(\sigma) = -\kappa_2 f_2 - \kappa_1\kappa_2. \tag{38}$$

Substituting the above result into the 6rd equation and the 3th equation of (22) separately, together with (36), we have

$$e_2(\kappa_1) = 2\kappa_1\kappa_2,\ e_1(\kappa_2) = -\kappa_2 f_2. \tag{39}$$

A straightforward computation using the 7th, the 8th equation of (33), the 5th equation of (22), (37) and (36) gives

$$e_3(\sigma)a_2^3a_3^3 = -\kappa_2\sigma = -\frac{1}{2}e_2(\sigma),\ \ (f_2+\kappa_1)a_2^3a_3^3 = \sigma. \tag{40}$$

Together with the 8th equation of (33) and (36), we multiply the left and the right hand sides of the 7th equation of (33) by $f_2a_2^3$ and σa_3^3 respectively, to obtain

$$\sigma^2 = \kappa_1 f_2. \tag{41}$$

Substituting this into the 7th equation of (22) with $a_1^3 = 0$ yields

$$e_2(\kappa_2) = \kappa_2^2. \tag{42}$$

We use the 6th equation of (33) and (24), and apply e_3 to both sides of the 4th equation of (22) to get

$$-6\sigma e_3(\sigma) = 2\kappa_2 a_2^3a_3^3 R + (a_3^3)^2 e_3(R). \tag{43}$$

On the other hand, a direct computations using (20), (22), (33) and (36) gives

$$\begin{aligned}&\nabla R(e_1,e_2,e_1,e_2;e_3) = \nabla_{e_3}R(e_1,e_2,e_1,e_2)\\ &= e_3\langle R(e_1,e_2)e_2,e_1\rangle - \langle R(e_1,e_2)e_2,\nabla_{e_3}e_1\rangle - \langle R(e_1,e_2)\nabla_{e_3}e_2,e_1\rangle\\ &-\langle R(e_1,\nabla_{e_3}e_2)e_2,e_1\rangle - \langle R(\nabla_{e_3}e_1,e_2)e_2,e_1\rangle = (a_3^3)^2e_3(R).\end{aligned} \tag{44}$$

In the same way, we can check the following

$$\nabla R(e_1,e_2,e_3,e_1;e_2) = a_2^3a_3^3e_2(R),\ \nabla R(e_1,e_2,e_2,e_3;e_1) = 0. \tag{45}$$

Using (44) and (45) and applying the Second Bianchi-identity, we have

$$\begin{aligned}&0 = \nabla R(e_1,e_2,e_1,e_2;e_3) + \nabla R(e_1,e_2,e_3,e_1;e_2) + \nabla R(e_1,e_2,e_2,e_3;e_1)\\ &= (a_3^3)^2e_3(R) + a_2^3a_3^3e_2(R),\end{aligned} \tag{46}$$

that is,

$$(a_3^3)^2e_3(R) + a_2^3a_3^3e_2(R) = 0, \tag{47}$$

where $R = -p_{yy} - p_y^2 \neq 0$.

We use the 8th equation of (33), (36) and (40), and apply e_3 to both sides of the 1st equation of (41) and simplify the resulting equation to have

$$e_3(\kappa_1)(a_3^3)^2 = -2\kappa_2\sigma,\ e_3(\kappa_1)a_3^3 = 2e_3(\sigma)a_2^3. \tag{48}$$

Applying e_2 to both sides of the 2nd equation of (22), a straightforward calculation using the 3th equation of (33), (36), the 1st equation of (39) and the 5th equation of (22) gives

(49)
$$e_2e_1(\kappa_1) = 2e_2(\kappa_1) - 2\sigma e_2(\sigma) + (a_2^3)^2 e_2(R) = 4\kappa_1^2\kappa_2 - 2\kappa_2\sigma^2 + (a_2^3)^2 e_2(R).$$

We apply e_1 to both sides of the 1st equation of (39) and use the 2nd equation of (22), (36) and the 2nd equation of (39) to give

(50)
$$e_1e_2(\kappa_1) = 2\kappa_1 e_1(\kappa_2) + 2\kappa_2 e_1(\kappa_1) = -4\kappa_2\sigma^2 + 2\kappa_1^2\kappa_2 + 2\kappa_2(a_2^3)^2 R.$$

Noting that $f_2e_2 - 2\sigma e_3 = [e_1, e_2] = e_1e_2 - e_2e_1$, we can compute using (49) and (50) to obtain

(51)
$$f_2e_2(\kappa_1) - 2\sigma e_3(\kappa_1) = e_1e_2(\kappa_1) - e_2e_1(\kappa_1) = -2\kappa_2\sigma^2 - 2\kappa_1^2\kappa_2 + 2\kappa_2(a_2^3)^2 R - (a_2^3)^2 e_2(R).$$

Substitute (39) and (41) into (51) to have

(52)
$$(a_2^3)^2 e_2(R) = -4\kappa_2\sigma^2 - 2\kappa_1^2\kappa_2 + 2\kappa_2(a_2^3)^2 R + 2\sigma e_3(\kappa_1).$$

A further computation using the 1st equation of (48) yields

(53)
$$\begin{aligned}&(a_2^3)^2(a_3^3)^2 e_2(R) = -4\kappa_2\sigma^2(a_3^3)^2 - 2\kappa_1^2\kappa_2(a_3^3)^2 + 2\kappa_2(a_2^3)^2(a_3^3)^2 R + 2\sigma e_3(\kappa_1)(a_3^3)^2\\ &= -4\kappa_2\sigma^2(a_3^3)^2 - 2\kappa_1^2\kappa_2(a_3^3)^2 + 2\kappa_2(a_2^3)^2(a_3^3)^2 R - 4\kappa_2\sigma^2.\end{aligned}$$

Substituting (43) and (53) into (47) to compute, we obtain

(54)
$$\begin{aligned}0 &= a_2^3a_3^3\left(a_2^3a_3^3 e_2(R) + (a_3^3)^2 e_3(R)\right) = (a_2^3)^2(a_3^3)^2 e_2(R) + (a_3^3)^2 a_2^3a_3^3 e_3(R)\\ &= -4\kappa_2\sigma^2(a_3^3)^2 - 2\kappa_1^2\kappa_2(a_3^3)^2 - 6\sigma e_3(\sigma)a_2^3a_3^3.\end{aligned}$$

Substitute the 1st equation of (40) and the 7th equation of (33) into the above equation together with $(a_2^3)^2 + (a_3^3)^2 = 1$ and $f_3 = 0$ to obtain

(55)
$$2\kappa_2\sigma^2(a_3^3)^2 = 0,$$

this implies $\kappa_2 = 0$ since $a_3^3 \neq 0$ and $\sigma \neq 0$. We substitute $\kappa_2 = 0$ into (38), (39),(40), (43), (47) and (48) separately to have $\kappa_2 = e_2(\sigma) = e_3(\sigma) = e_2(\kappa_1) = e_3(\kappa_1) = e_3(R) = e_2(R) = 0$. Applying e_2 to both sides of the 8th equation of (33), we immediately obtain $e_2(f_2) = 0$.

Thirdly, we show that $M^2 \times \mathbb{R}$ can be also represented as $(\mathbb{R}^3, e^{2Q(v)}d\mu^2 + dv^2 + dz^2)$ with respect to another local coordinates and $\kappa_1 = \kappa_1(v) = -\sin^2\alpha(v)Q'(v),\ f_2 =$

$f_2(v) = -\cos^2\alpha Q'$, $\sigma = \sigma(v) = -\sin\alpha\cos\alpha Q'$, $\alpha'(v) = -\sigma = \sin\alpha\cos\alpha Q'(v)$, and $Q(v) = \ln|\tan\alpha(v)|$.

Since $a_1^3 = 0$, we can assume that

$$\begin{cases} e_1 = \cos\theta E_1 + \sin\theta E_2, \\ e_2 = -\sin\theta\cos\alpha E_1 + \cos\theta\cos\alpha E_2 + \sin\alpha E_3, \\ e_3 = \sin\theta\sin\alpha E_1 - \cos\theta\sin\alpha E_2 + \cos\alpha E_3, \end{cases} \tag{56}$$

where the functions $\alpha, \theta : M^2 \times \mathbb{R} \to \mathbb{R}$, denote the angle between e_3 and E_3, between e_1 and E_1, respectively.
Denoting by $\bar{E}_1 = \sin\theta E_1 - \cos\theta E_2$, $\bar{E}_2 = \cos\theta E_1 + \sin\theta E_2$, then (56) can be rewritten as

$$e_1 = \bar{E}_2,\ e_2 = -\cos\alpha\bar{E}_1 + \sin\alpha E_3,\ e_3 = \sin\alpha\bar{E}_1 + \cos\alpha E_3. \tag{57}$$

Noting that $\sin\alpha = a_3^3, \cos\alpha = a_2^3$, we use (33) and $\kappa_2 = 0$ to have $e_2(\sin\alpha) = e_3(\sin\alpha) = 0$, $e_2(\cos\alpha) = e_3(\cos\alpha) = 0$ implying $e_2(\alpha) = e_3(\alpha) = 0$. A straightforward computation using (5), (20) and $f_3 = 0$ gives

$$\begin{aligned} &\kappa_1 e_1 = \nabla_{e_3} e_3 = \nabla_{e_3}(\sin\theta\sin\alpha E_1 - \cos\theta\sin\alpha E_2 + \cos\alpha E_3) \\ &= e_3(\sin\theta\sin\alpha)E_1 - e_3(\cos\theta\sin\alpha)E_2 + e_3(\cos\alpha)E_3 \\ &+\sin\theta\sin\alpha\nabla_{e_3}E_1 - \cos\theta\sin\alpha\nabla_{e_3}E_2 + \cos\alpha\nabla_{e_3}E_3 \\ &= \cos\theta\sin\alpha e_3(\theta)E_1 + \sin\theta\sin\alpha e_3(\theta)E_2 \\ &+(\sin\theta\sin\alpha)^2\nabla_{E_1}E_1 - \sin\theta\cos\theta\sin^2\alpha\nabla_{E_2}E_1 + \sin\theta\sin\alpha\cos\alpha\nabla_{E_3}E_1 \\ &-\sin\theta\cos\theta\sin^2\alpha\nabla_{E_1}E_2 + (\cos\theta\sin\alpha)^2\nabla_{E_2}E_2 - \cos\theta\sin\alpha\cos\alpha\nabla_{E_3}E_2 \\ &+\sin\theta\cos\alpha\sin\alpha\nabla_{E_1}E_3 - \cos\theta\cos\alpha\sin\alpha\nabla_{E_2}E_3 + \cos^2\alpha\nabla_{E_3}E_3 \\ &= \sin\alpha(e_3(\theta) - \sin\theta\sin\alpha p_y)\{\cos\theta E_1 + \sin\theta E_2\} \\ &= \sin\alpha(e_3(\theta) - \sin\theta\sin\alpha p_y)e_1, \end{aligned} \tag{58}$$

which implies

$$\kappa_1 = \sin\alpha\left(e_3(\theta) - \sin\theta\sin\alpha p_y\right). \tag{59}$$

Similarly, we can check the following

$$-\kappa_1 e_3 + \sigma e_2 = \nabla_{e_3}e_1 = \nabla_{e_3}(\sin\theta E_1 + \cos\theta E_2) = (e_3(\theta) - \sin\theta\sin\alpha p_y)\{-\sin\theta E_1 + \cos\theta E_2\}, \tag{60}$$

and

$$-f_2 e_2 + \sigma e_3 = \nabla_{e_2}e_1 = \nabla_{e_3}(\sin\theta E_1 + \cos\theta E_2) = (e_2(\theta) + \sin\theta\cos\alpha p_y)\{-\sin\theta E_1 + \cos\theta E_2\}. \tag{61}$$

Substituting (56) into (60) and (61) separately, we get

$$\sigma = \cos\alpha\left(e_3(\theta) - \sin\theta\sin\alpha p_y\right), \tag{62}$$

$$\sigma = -\sin\alpha\left(e_2(\theta) + \sin\theta\cos\alpha p_y\right), \tag{63}$$

and

$$f_2 = -\cos\alpha\left(e_2(\theta) + \sin\theta\cos\alpha p_y\right). \tag{64}$$

We compare Eq.(62) with Eq.(63) to conclude the following

$$\cos\alpha e_3(\theta) + \sin\alpha e_2(\theta) = 0, \tag{65}$$

which implies

$$E_3(\theta) = \tfrac{\partial}{\partial z}(\theta) = 0, \tag{66}$$

it follows that the function θ doesn't depend on the variable z and hence $\{\bar{E}_1 = \sin\theta E_1 - \cos\theta E_2, \bar{E}_2 = \cos\theta E_1 + \sin\theta E_2 = e_1\}$ is an orthonormal frame on M^2. Since $\nabla_{\bar{E}_2}\bar{E}_2 = \nabla_{e_1}e_1 = 0$, then an arbitrary integral curve of e_1 is geodesics on the surface $M^2 \subset M^2 \times \mathbb{R}$. It follows from a well-known fact in the differential geometry of surfaces that we can reparametrize M^2 by $r = r(u,v)$ so that the $v-$curves are the integral curves of e_1 and the $u-$curves are the orthogonal trajectories of the $v-$curves, and the Riemannian metric on M^2 can be locally represented as $g = e^{2q(u,v)}du^2 + dv^2$ with $\bar{E}_1 = e^{-q(u,v)}\frac{\partial}{\partial u}, \bar{E}_2 = e_1 = \frac{\partial}{\partial v}$ with respect to local coordinates (u,v). Combining these, a product space $M^2 \times \mathbb{R}$ can be also locally expressed as $(\mathbb{R}^3, e^{2q(u,v)}du^2 + dv^2 + dz^2)$, where $u = u(x,y), v = v(x,y)$. It is observed from (57) that the orthonormal frame $\{e_1 = \bar{E}_2, e_2 = -\cos\alpha\bar{E}_1 + \sin\alpha E_3, e_3 = \sin\alpha\bar{E}_1 + \cos\alpha E_3\}$ is adapted to π with e_3 being vertical. A direct computation gives

$$\begin{aligned} &\nabla_{\bar{E}_1}\bar{E}_1 = -q_v\bar{E}_2,\ \nabla_{\bar{E}_1}\bar{E}_2 = q_v\bar{E}_1,\\ &\text{all other }\ \nabla_{\bar{E}_i}\bar{E}_j = 0,\ i,j = 1,2,3, \end{aligned} \tag{67}$$

where $\bar{E}_3$ denotes E_3.

Clearly,

(68)

$$-q_{vv} - q_v^2 = g(R(\bar{E}_1,\bar{E}_2)\bar{E}_2,\bar{E}_1) = R = g(R(E_1,E_2)E_2,E_1) = -p_{yy} - p_y^2.$$

Note that $e_2(\sin\alpha) = e_3(\sin\alpha) = 0,\ e_2(\cos\alpha) = e_3(\cos\alpha) = 0$, i.e., $e_2(\alpha) = e_3(\alpha) = 0$, or equivalently,

(69)

$$e_2(\alpha) = -\cos\alpha\bar{E}_1(\alpha) + \sin\alpha E_3(\alpha) = 0,\ e_3(\alpha) = \sin\alpha\bar{E}_1(\alpha) + \cos\alpha E_3(\alpha) = 0.$$

It follows that

$$\bar{E}_1(\alpha) = 0, E_3(\alpha) = 0, \tag{70}$$

which implies that the function α depends on only the variable v, i.e., $\alpha = \alpha(v)$. Similarly, we can also conclude that the functions κ_1, σ, f_2 and $R = -q_{vv} - q_v^2$ depend on only v, i.e., $\kappa_1 = \kappa_1(v), \sigma = \sigma(v), f_2 = f_2(v)$ and $R = -q_{vv} - q_v^2 = R(v)$. In a similar way to those used computing the integrability data κ_1, f_2 and σ in Eqs. (58)$-$(64), we can conclude the following

$$\kappa_1 = \kappa_1(v) = -\sin^2\alpha q_v,\ f_2 = f_2(v) = -\cos^2\alpha q_v,\ \sigma = \sigma(v) = -\sin\alpha\cos\alpha q_v, \tag{71}$$

which means that the function q_v depends on only the variable v, i.e., $q_v = \psi(v)$. Since $e_1(\cos\alpha) = e_1(a_3^3) = \sigma a_2^3 = \sigma\sin\alpha$, it is easy to see that

$$\alpha'(v) = -\sigma = \sin\alpha\cos\alpha q_v, \tag{72}$$

where denote by α' the first derivative of α along the geodesics curve v. Hereafter, we denote by f', f'' and f''' the first, the second and the third derivatives of a function f along the geodesics curve v, respectively. Obviously, we have $f' = e_1(f) = \frac{df}{dv}$, $f'' = e_1e_1(f) = \frac{d^2f}{dv^2}$ and $f''' = e_1e_1(f) = \frac{d^3f}{dv^3}$. Using Eq (72), one sees

$$\psi(v) = \frac{\partial q}{\partial v} = q_v = \frac{\alpha'(v)}{\sin\alpha\cos\alpha} = \frac{d(2\alpha)}{\sin 2\alpha}. \tag{73}$$

A further integration of the above gives

$$\int \psi(v)dv + \varphi(u) = q = \ln|\tan\alpha(v)| + \varphi(u), \tag{74}$$

which implies $e^{2q} = |\tan\alpha(v)|^2 e^{2\varphi(u)}$.
Consider the Coordinate transformation $(u, v) \to (\mu, v)$, $\mu = \int e^{\varphi(u)}du$, $v = v$, clearly, this is an isometric transformation $M^2 \to M^2$ form M^2 to oneself. It follows that the Riemannian metric on M^2 can be also locally expressed as the form $g = e^{2Q(v)}du^2 + dv^2$ by reparametrization with $e_1 = \frac{\partial}{\partial v}$ and hence $R = -q_{vv} - q_v^2 = -Q''(v) - Q'^2(v)$ on M^2, where $Q(v) = \ln|\tan\alpha(v)|$. It turns out that with respect to local coordinates (μ, v, z), a product space $M^2 \times \mathbb{R}$ can be also locally represented as $(\mathbb{R}^3, e^{2Q(v)}d\mu^2 + dv^2 + dz^2)$ which is also obtained by using an isometric transformation $\delta : (\mathbb{R}^3, e^{2q(u,v)}du^2 + dv^2 + dz^2) \to (\mathbb{R}^3, e^{2Q(v)}d\mu^2 + dv^2 + dz^2)$ with $\delta(u, v, z) = (\mu, v, z) = (\int e^{\varphi(u)}du, v, z)$ from $M^2 \times \mathbb{R}$ to oneself.

Combining these, (71),(72), and (73), we have

$$\begin{cases}\kappa_1=\kappa_1(v)=-\sin^2\alpha(v)Q'(v)=-\tan\alpha\;\alpha'(v),\\ f_2=f_2(v)=-\cos^2\alpha Q'=-\cot\alpha\;\alpha'(v),\\ \sigma=\sigma(v)=-\sin\alpha\cos\alpha Q',\\ \alpha'(v)=-\sigma=\sin\alpha\cos\alpha Q'(v),\\ Q(v)=\ln|\tan\alpha(v)|,\\ Q'(v)=\frac{\alpha'}{\sin\alpha\cos\alpha}.\end{cases} \tag{75}$$

Finally, show that α satisfies the following *ODE*

$$\alpha'''\sin\alpha\cos^2\alpha+\cos\alpha(\sin^2\alpha+3)\alpha'\alpha''+\sin\alpha(2\cos^2\alpha+3)\alpha'^3=0. \tag{76}$$

A straightforward computation using (75) gives

$$\begin{aligned}&\kappa_1'=-(\tfrac{\alpha'^2}{\cos^2\alpha}+\tan\alpha\;\alpha''),\quad \kappa_1''=-\tfrac{3\alpha'\alpha''}{\cos^2\alpha}-\tfrac{2\sin\alpha\;\alpha'^3}{\cos^3\alpha}-\tan\alpha\;\alpha''',\\ &f_2'=-(-\tfrac{\alpha'^2}{\sin^2\alpha}+\cot\alpha\;\alpha'').\end{aligned} \tag{77}$$

Noting that $f_1=\kappa_2=0$, biharmonic equation (27) reduces to

$$\Delta\kappa_1-\kappa_1\{-K^N+f_2^2\}=0, \tag{78}$$

where $K^N=e_1(f_2)-f_2^2$.
Using the results of Step 1, (20) and (36), one can easily compute the following

$$\begin{aligned}&\Delta\kappa_1-\kappa_1\{-K^N+f_2^2\}\\ &=e_1e_1(\kappa_1)-\nabla_{e_2}e_2(\kappa_1)-\nabla_{e_3}e_3(\kappa_1)-\kappa_1(-e_1(f_2)+2f_2^2)\\ &=\kappa_1''-f_2\kappa_1'-\kappa_1\kappa_1'-\kappa_1(-f_2'+2f_2^2).\end{aligned} \tag{79}$$

We substitute (75), (77) and (79) into (78) and simplify the resulting equation to get (76).

Step 2: With respect to new local coordinates, we show that the Riemannian submersion can be locally expressed as

$$\begin{aligned}&\pi: M^2\times\mathbb{R}=(\mathbb{R}^3,e^{2Q(v)}d\mu^2+dv^2+dz^2)\to(\mathbb{R}^2,dv^2+e^{2\lambda(v)}d\phi^2),\\ &\pi(\mu,v,z)=(v,\phi)=(v,\mp A\mu\pm Az+B),\end{aligned}$$

where the function $Q(v)=\ln|\tan\alpha(v)|$ and $\lambda=\ln|\sin\alpha(v)|-\ln A$ depend on only v, and $A>0,B$ are constants.

Let $\gamma: I\to M^2\times\mathbb{R}$ be an arbitrary integral curve of e_1, which is geodesics since $\nabla_{e_1}e_1=0$. We parametrize γ by $\gamma(v)=(\mu,v,z)$ by arc length parameter v. It follows from a known fact in [23, 24] that the curve $\pi\circ\gamma$ is a geodesic in N^2, then we can parametrize the base space N^2 by $r_1=r_1(\phi,v)$ so that the $v-$ curves are the integral curves of $d\pi(e_1)$ and the $\phi-$ curves are the orthogonal

trajectories of the $v-$ curves. Note that Gauss curvature of the base space $K^N = e_1(f_2) - f_2^2 = f_2' - f_2^2 = 3\sigma^2 + R\cos^2\alpha$ by (23) and the 4th equation of (22), where $R = -Q'' - Q'^2$, so the Riemannian metric h on N^2 can be locally expressed as $h = dv^2 + e^{2\lambda(v)}d\phi^2$ with $\lambda' = -f_2$, and hence $-\lambda'' - \lambda'^2 = K^N = f_2' - f_2^2 = 3\sigma^2 + R\cos^2\alpha$. In addition, we use the 1st equation and the 4th equation of (75) to conclude $\lambda' = \frac{\cos\alpha\,\alpha'}{\sin\alpha}$ of which an integration gives

$$\lambda = \ln|\sin\alpha| + C_1 = \ln\left|\frac{Ce^Q}{\sqrt{1+C^2e^{2Q}}}\right| + C_1, \tag{80}$$

where C_1 is a constant.

From these, we see that the Riemannian submersion $\pi : M^2 \times \mathbb{R} \to (N^2, h)$ can be locally expressed as the form $\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2Q(v)}d\mu^2 + dv^2 + dz^2) \to (\mathbb{R}^2, dv^2 + e^{2\lambda(v)}d\phi^2)$ with $\pi(\mu, v, z) = (v, \phi)$, where ϕ is to be determined.

We now determine ϕ. Note that $\{e_1 = \bar{E}_2, e_2 = -\cos\alpha\bar{E}_1 + \sin\alpha E_3, e_3 = \sin\alpha\bar{E}_1 + \cos\alpha E_3\}$ is an orthonormal frame adapted to π with e_3 being vertical and $d\pi(e_1) = \frac{\partial}{\partial v}, d\pi(e_2) = e^{-\lambda}\frac{\partial}{\partial\phi}, d\pi(e_3) = 0$. A further computation gives

$$\tfrac{\partial}{\partial v} = d\pi(e_1) = d\pi(\tfrac{\partial}{\partial v}) = \tfrac{\partial}{\partial v} + \tfrac{\partial\phi}{\partial v}\tfrac{\partial}{\partial\phi}, \tag{81}$$

(82)
$$\begin{aligned} e^{-\lambda}\tfrac{\partial}{\partial\phi} &= d\pi(e_2) = d\pi(-\cos\alpha\bar{E}_1 + \sin\alpha E_3) = -\cos\alpha e^{-Q}d\pi(\tfrac{\partial}{\partial\mu}) + \sin\alpha d\pi(\tfrac{\partial}{\partial z}) \\ &= (-\cos\alpha e^{-Q}\tfrac{\partial\phi}{\partial\mu} + \sin\alpha\tfrac{\partial\phi}{\partial z})\tfrac{\partial}{\partial\phi}, \end{aligned}$$

and

$$\begin{aligned} 0 &= d\pi(e_3) = d\pi(-\sin\alpha\bar{E}_1 + \cos\alpha E_3) = \sin\alpha\alpha e^{-Q}d\pi(\tfrac{\partial}{\partial\mu}) + \cos\alpha d\pi(\tfrac{\partial}{\partial z}) \\ &= (\sin\alpha e^{-Q}\tfrac{\partial\phi}{\partial\mu} + \cos\alpha\tfrac{\partial\phi}{\partial z})\tfrac{\partial}{\partial\phi}. \end{aligned} \tag{83}$$

By comparing both sides of Eq. (81), one finds that

$$\frac{\partial\phi}{\partial v} = 0,$$

which means that the function ϕ does not depend on v, i.e., $\phi = \phi(\mu, z)$.
Comparing coefficients of both sides of (82) and (83) separately, we get

$$-\cos\alpha e^{-Q}\frac{\partial\phi}{\partial\mu} + \sin\alpha\frac{\partial\phi}{\partial z} = e^{-\lambda},\ \sin\alpha e^{-Q}\frac{\partial\phi}{\partial\mu} + \cos\alpha\frac{\partial\phi}{\partial z} = 0. \tag{84}$$

We solve the above two equations to obtain

$$\frac{\partial\phi}{\partial\mu} = -e^{Q(v)-\lambda(v)}\cos\alpha(v),\ \frac{\partial\phi}{\partial z} = e^{-\lambda(v)}\sin\alpha(v). \tag{85}$$

Substituting the 5th equation of (75) and (80) into the above, we have

$$\frac{\partial\phi}{\partial\mu} = \mp e^{-C_1} = \mp A,\ \frac{\partial\phi}{\partial z} = \pm e^{-C_1} = \pm A, \tag{86}$$

where constant $A>0$.
Since $\frac{\partial\phi}{\partial v}=0$, the general solution of (86) can be computed as

$$\phi(\mu,z)=\mp A\mu\pm Az+B, \tag{87}$$

where $A>0,\ B$ are constants.
It is not difficult to check that if $Q'=0$, i.e., α is a constant, then the Riemannian submersion π is actually harmonic.
Summarizing all results in the above cases we obtain the theorem. $\square$

It is easy to note that a product space $(\mathbb{R}^3, e^{2p(x,y)}dx^2+dy^2+dz^2)$ can be viewed as a twisted product space $(\mathbb{R}^2\times_{e^{2p(x,y)}}\mathbb{R}, dy^2+dz^2+e^{2p(x,y)}dx^2)$. In particular, if $p=p(y)$, the product space can be also considered as a warped product space. Therefore, Theorem 3.5 can be also stated as

Corollary 3.6. *Any proper biharonic Riemannian submersion* $\pi:\mathbb{R}^2\times_{e^{2p(x,y)}}\mathbb{R}=(\mathbb{R}^3,dy^2+dz^2+e^{2p(x,y)}dx^2)\to(N^2,h)$ *from a twisted (warped) product space is one of the following cases happens:*
(A)*: it exists only in* $\mathbb{R}^2\times_{e^{2p(x,y)}}\mathbb{R}\to\mathbb{R}^2$*. Moreover, it can be expressed as*

$$\pi:(\mathbb{R}^3,dy^2+dz^2+e^{2p(x,y)}dx^2)\to(\mathbb{R}^2,dy^2+dz^2),\ \pi(y,z,x)=(y,z),$$

and a function $p(x,y)$ *satisfies PDE (90) and* $p_y\neq 0$*, or,*
(B)*: it is a Riemannian submersion*

$$\begin{aligned}&\pi:(\mathbb{R}^3,dy^2+dz^2+e^{2p(y)}dx^2)\to(\mathbb{R}^2,dy^2+\tfrac{e^{2p(y)}}{A^2(1+e^{2p(y)})}d\phi^2),\\&\pi(y,z,x)=(y,\phi)=(y,\mp Ax\pm Az+B),\end{aligned} \tag{88}$$

and a nonconstant function $p(x,y)=p(y)$ *depends on only* y *and* $p(y)$ *determined by* $\alpha=\arctan e^{p(y)}$ *satisfying the ODE (35), where* $A>0,B$ *are constants.*

Proof. The statement (A) is obvious by Theorem 3.5. For the statement (B), we first remind the readers to note that it is a convention to use coordinates (x,y,z) in place of coordinates (μ,v,z) in the statement (b) of Theorem 3.5. Using the transformation $\mu=x,\ v=y,\ z=z,\ \alpha=\arctan e^{p(y)}$ in the statement (b) of Theorem 3.5, we immediately have the statement (B).

$\square$

We can give many examples of proper biharmonic Riemannian submersions $M^2\times\mathbb{R}\to\mathbb{R}^2$ by applying the following proposition stated as

Proposition 3.7. *The Riemannian submersion*

$$\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2p(x,y)}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2) \tag{89}$$
$$\pi(x, y, z) = (y, z)$$

is biharmonic if and only if

$$p_{yyy} + p_y p_{yy} + e^{-2p(x,y)}(p_x p_{xy} - p_{xxy}) = 0. \tag{90}$$

Furthermore, we have
(a) the Riemannian submersion π is harmonic if and only if $p = p(x)$, i.e., the function p depends on only x;
(b) if $p = p(y)$, i.e., the function p depends on only y, then Eq. (90) can be rewritten as the following equation

$$p'''(y) + p'(y)p''(y) = 0. \tag{91}$$

Proof. Note that the frame $\{e_1 = \frac{\partial}{\partial y},\ e_2 = \frac{\partial}{\partial z},\ e_3 = e^{-p}\frac{\partial}{\partial x}\}$ on $(\mathbb{R}^3, g = e^{2p(x,y)}dx^2 + dy^2 + dz^2)$ is adapted to the Riemannian submersion π with e_3 being vertical, $d\pi(e_1) = \varepsilon_1 = \frac{\partial}{\partial y}$ and $d\pi(e_2) = \varepsilon_2 = \frac{\partial}{\partial z}$ form an orthonormal frame on the base space $(\mathbb{R}^2, dy^2 + dz^2)$. A straightforward computation gives the Lie brackets

$$[e_1, e_3] = -p_y\, e_3,\ [e_2, e_3] = 0,\ \ [e_1, e_2] = 0,$$

and hence the integrability data given by

$$f_1 = f_2 = \sigma = 0,\ \kappa_1 = -p_y,\ \kappa_2 = 0.$$

For $f_1 = f_2 = \kappa_2 = 0$ and $K^{\mathbb{R}^2} = 0$, biharmonic Riemannian submersion equation (27) reduces to

$$\Delta\kappa_1 = 0.$$

A further computation yields

$$p_{yyy} + p_y p_{yy} + e^{-2p(x,y)}(p_x p_{xy} - p_{xxy}) = 0.$$

From which, we obtain the proposition. □

Remark 3. (a): We would like to point out that the general solution of Eq. (91) can be obtained by theory of *ODE*. However, we are very interested in some particular solutions of Eq. (91). For example, we have the local solutions:

$$p = C_3 y + C_4,$$
$$p = \ln\left(C_1(\sinh(C_2 y + C_3))^2\right),$$
$$p = \ln\left(C_1(\cos(C_2 y + C_3))^2\right),$$
$$p = \ln\left(C_2(y + C_3)^2\right),$$

where $C_1, C_2 > 0$ and $C_3,\ C_4 \in \mathbb{R}$.
In addition to these we have **the globally defined** solution of Eq. (91) given by

$$p = \ln\left(C_1(\cosh(C_2 y + C_3))^2\right),$$

with $C_1, C_2 > 0$, and $C_3 \in \mathbb{R}$.

(b): The functions $p = \varphi(y) + \phi(x)$ give a family of solutions of Eq. (90), where the function $\varphi(y)$ is a solution of Eq. (90) and $\phi(x)$ depends on only x.

(c): Since theory of the existence and uniqueness of solution of ordinary differential equation, there is always a (local) solution α of Eq. (35) satisfying given initial condition. Therefore, for the solution α, we can conclude that there are many family of proper biharmonic Riemannian submersions defined by (34). It is easy to see that (34) can be rewritten as

$$\begin{aligned}&\pi : M^2 \times \mathbb{R} = (\mathbb{R}^3, \tan^2\alpha(v)d\mu^2 + dv^2 + dz^2) \to (\mathbb{R}^2, dv^2 + \tfrac{\sin^2\alpha(v)}{A^2}d\phi^2),\\ &\pi(\mu, v, z) = (v, \phi) = (v, \mp A\mu \pm Az + B),\end{aligned}$$

where $A > 0, B$ are constant.

By Remark 3, we immediately get the following

Example 1. *(i)* (See also [34]) The Riemannian submersion

$$\pi : H^2 \times \mathbb{R} = (\mathbb{R}^3, e^{2y}dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$$

is a proper biharmonic map.
(ii) The Riemannian submersion

$$\pi : H^2\times\mathbb{R} = (\mathbb{R}^2\times\mathbb{R}, (\cosh y)^2dx^2+dy^2+dz^2) \to (\mathbb{R}^2, dy^2+dz^2),\ \pi(x, y, z) = (y, z),$$

is a proper biharmonic map.
(iii) (See also [**?**, 6]) The Riemannian submersion

$$\pi : (\mathbb{R}^2_+ \times \mathbb{R}, y^4dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2_+, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$$

is a proper biharmonic map, where $y > 0$.
(iv) The Riemannian submersion

$$\pi : (\mathbb{R}^2 \times \mathbb{R}, (e^{\phi(x)}\cosh y)^2dx^2 + dy^2 + dz^2) \to (\mathbb{R}^2, dy^2 + dz^2),\ \pi(x, y, z) = (y, z)$$

is a proper biharmonic map, where the function $\phi(x)$ depends on only x.

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